

The generalized second law of thermodynamics for interacting $f(R)$ gravity

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(Dated: June 25, 2014)

Abstract

We examine the validity of the generalized second law (GSL) of gravitational thermodynamics in the context of interacting $f(R)$ gravity. We take into account that the boundary of the universe to be confined by the dynamical apparent horizon in a flat FRW universe. We study the effective equation of state, deceleration parameter and GSL in this interaction-framework. We find that the evolution of the total entropy increases through the interaction term. As a example, we consider a $f(R)$ gravity with a power-law dependence on the curvature R . Here, we find exact solutions for a model in which the interaction term is related to the total energy density of matter.

PACS numbers: 98.80.Cq

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I. INTRODUCTION

The observational data of the luminosity-redshift of type Ia supernovae (SNeIa), large scale structure (LSS) and the cosmic microwave background (CMB) anisotropy spectrum, have supported evidence that our universe has recently arrived a phase of accelerated expansion [1–6]. For this current acceleration a possible responsible is the dark energy (DE) and the nature of this DE is a problem today. For a review of DE models, see Refs. [7–12].

In the last years, a $f(R)$ theory was proposed to elucidate the expansion of the universe without taking the DE [13–15]. In such an approach, the Ricci scalar R in the Einstein-Hilbert action is replaced by a general function $f(R)$, (for a review see Refs. [16–19]). Also, there are other classes of modified gravities that can give an explication for the different cosmological scenarios without take into account to the DE. In particular, $f(T)$ theory that is a generalization of the teleparallel gravity (TG) and becomes equivalent of General Relativity [20–22]. Also, the modified gravity that includes the Gauss-Bonnet invariant term $f(\mathcal{G})$ [23–29]. In this context, there are two forms to analyze dark energy energy models, by means of a fluid explanation and the other is to define the action associate to a scalar field (see e.g. Ref.[30]). In particular, for the background solutions these two ways to describe dark energy models are equivalent. However, one cannot assume univocally this equivalence, for example in the stability of the solutions[31] or in the studies of cosmological perturbations[13, 32].

On the other hand, in the context of the thermodynamics point of view, the accelerating universe has conceived much consideration and different types of consequences has been detected [33, 34]. Specifically, the confirmation of the first and second law of the thermodynamics, from a dynamic aspect together with a thermodynamic analysis of the accelerating universe.

For the validity of the generalized second law (GSL) of thermodynamics, is essential that the evolution with respect to the cosmic time of the total entropy \dot{S}_{Total} , becomes $\dot{S}_{Total} = d(S_A + S_m)/dt \geq 0$. S_A is the Bekenstein-Hawking entropy on the apparent horizon and S_m represents the entropy of the universe filled with matter [35]. Consequently, in accordance with the GSL of thermodynamics, the development of the total entropy, S_{Total} , cannot decrease in the time [36–41].

In the frame of reference of the first law of thermodynamics, we can write for the ap-

parent horizon $-dE = T_A dS_A$ and obtain in this form the Einstein's field equation. Also, the agreement between the first law of thermodynamics and the Einstein's field equation is fulfilled, if we take into account that the Hawking temperature T_A , in which $T_A \propto R_A^{-1}$ and also the entropy on the apparent horizon $S_A \propto A$, in which R_A and A are the radius and area related to the horizon [35], see also Ref. [42–44]. Nevertheless, it should also be note that the entropy on the apparent horizon, is modified for other types of theories. Specifically, in $f(R)$ gravity, the geometric entropy is given by $S_A = Af_R/4G$ [45], where $f_R = \partial f(R)/\partial R$. Also, in the context of the model $f(T)$ gravity, in Ref. [46], the authors calculated that when f'' is small, the entropy of the apparent horizon is given by $S_A = Af'/4G$, here the primes denote derivative with respect to the torsion scalar T , see also Ref.[47].

In the framework of $f(R)$ gravity the GSL of thermodynamics, was studied in Ref. [48] (see also Ref. [49–51]). In this work, the authors analyzed a Friedmann-Robertson-Walker (FRW) universe filled only with ordinary matter enclosed by S_A and examined the validity of the GSL for a viable $f(R)$ model; $f(R) = R - \alpha/R + \beta R^2$.

On the other hand, to solve the cosmic coincidence problem [8, 52], several authors have analyzed the interaction between DE and DM components [53–56]. Here, the interaction term can mitigate the coincidence problem in the sense that the rate between both densities either leads to a constant or changes slowly in late times [57, 58]. In connection with the GSL, the analysis of the validity of the GSL in the presence of an interaction between DM and DE was studied in Ref. [59]. In this model, the authors considered that the interaction between both component is proportional to the DE. Additionally, the thermodynamic description for the interaction between holographic DE and DM was considered in Ref. [60] and also an analysis of the GSL for the interacting generalized Chaplygin gas model was studied in [61]. In the context of the interaction between DE and DM from the Le Châtelier-Braun principle was analyzed in Ref. [62].

The goal of this work is to study the validity of the GSL of thermodynamics considering the interacting $f(R)$ gravity model. We will analyze a flat universe FRW background filled with the pressureless matter. Also, we study the equation of state (EoS) of the model, the deceleration parameter, and the GSL of gravitational thermodynamics. Finally as an example, we analyzed a $f(R)$ model together with a particular interaction term.

The outline of the paper is as follows. The next section presents the interacting $f(R)$ gravity in a flat FRW universe. Here, we investigate the EoS and the deceleration parameter.

Section III we study the validity of the GSL of thermodynamics in the context of the interacting $f(R)$ gravity. Section IV we analyze an example for $f(R)$ and a particular interaction term Q . Section V we study the conformal transformation and the GSL in a scalar tensor gravity theory. Finally, in Sect. VI we summarize our finding. We chose units such that $c = \hbar = 8\pi G = 1$.

II. INTERACTING $f(R)$ GRAVITY

The action I in the framework of $f(R)$ gravity, becomes [17, 63, 64]

$$I = \int d^4x \sqrt{-g} \left[\frac{f(R)}{2} + L_m \right]. \quad (1)$$

Here L_m is related to the Lagrangian density of the matter inside the universe.

In order to describe the $f(R)$ theory we start with the following gravitational field equations in a flat FRW background filled with the pressureless matter

$$H^2 = \frac{1}{3} \rho_t, \quad (2)$$

$$\dot{H} = -\frac{1}{2} (\rho_t + p_t), \quad (3)$$

where ρ_t and p_t are the total energy density and pressure given by

$$\rho_t = \frac{\rho_m}{f_R} + \rho_R, \quad \text{and} \quad p_t = p_R. \quad (4)$$

Here, ρ_R and p_R are the energy density and pressure due to the curvature contribution, defined as [17, 63, 64]

$$\rho_R = \frac{1}{f_R} \left(-\frac{1}{2} (f - Rf_R) - 3H\dot{f}_R \right), \quad (5)$$

and

$$p_R = \frac{1}{f_R} \left(\frac{1}{2} (f - Rf_R) + 2H\dot{f}_R + \ddot{f}_R \right), \quad (6)$$

and the energy density of the matter ρ_m , is given by

$$\rho_m = \frac{f}{2} - 3 \left(\dot{H} + H^2 - H \frac{d}{dt} \right) f_R, \quad (7)$$

where, $H = \dot{a}/a$ is the Hubble factor, a is a scale factor, $R = 6 \left(\dot{H} + 2H^2 \right)$ is the scalar curvature and $f_R = \partial f(R)/\partial R$. Dots here mean derivatives with respect to the cosmological time.

On the other hand, we shall consider that both components, i.e., the scalar curvature and the cold dark matter do not conserve separately but that they interact through a Q term (to be specified later) according to

$$\dot{\rho}_m + 3H\rho_m = Q, \quad (8)$$

and

$$\dot{\rho}_R + 3H(\rho_R + p_R) - \frac{\dot{f}_R}{f_R^2}\rho_m = -Q. \quad (9)$$

Note that the energy conservation law for the total perfect fluid is $\dot{\rho}_t + 3H(\rho_t + p_t) = 0$. In what follows we shall assume $Q > 0$. We also consider that the curvature contribution component obeys an equation of state (EoS) parameter $w_R = p_R/\rho_R$ and then the Eq.(9), becomes

$$\dot{\rho}_R - \frac{\dot{f}_R}{f_R^2}\rho_m + 3H\rho_R \left(1 + w_R + \frac{Q}{3H\rho_R}\right) = 0. \quad (10)$$

Taking time derivative of Eq.(5), we obtain

$$\begin{aligned} \dot{\rho}_R = & -\frac{\dot{f}_R}{f_R^2} \left(-\frac{1}{2}(f - Rf_R) - 3H\dot{f}_R \right) \\ & + \frac{1}{f_R} \left(\frac{R\dot{f}_R}{2} - 3\dot{H}\dot{f}_R - 3H\ddot{f}_R \right). \end{aligned} \quad (11)$$

From Ref.[61], we combining Eq.(10) and (11) and the EoS parameter results

$$w_R = - \left[1 + \frac{Q}{3H\rho_R} + \left(\frac{(\ddot{f}_R - H\dot{f}_R)}{[(f - Rf_R)/2 + 3H\dot{f}_R]} \right) \right], \quad (12)$$

here, we noted that the Eq.(12) corresponds to an effective EoS parameter.

On the other hand, the deceleration parameter q is defined as $q = - \left[1 + \frac{\ddot{H}}{H^2} \right]$, and considering Eqs.(2) and (3), yields

$$q = \frac{1}{2} \left[1 + \frac{\rho_R w_R}{H^2} \right]. \quad (13)$$

Combining Eqs.(12) and (13) the deceleration parameter q can be written as

$$\begin{aligned} q = & \frac{1}{2} + \frac{1}{2H^2 f_R} \left(\frac{1}{2}(f - Rf_R) + 3H\dot{f}_R \right) \\ & - \frac{1}{6H^3} \left(Q - \frac{3H(\ddot{f}_R - H\dot{f}_R)}{f_R} \right). \end{aligned} \quad (14)$$

We noted that for the particular case in which non-interacting limit $Q = 0$ and $f(R) = R$ the Eq.(14) results in $q = 1/2$, representing to the matter dominated epoch.

III. GSL INTERACTING - $f(R)$

It is well known that for the GSL, the entropy of the horizon plus the entropy of the matter within the horizon cannot decrease in time, see Refs.[36–41]. We consider that the boundary of the universe to be enclosed by the dynamical apparent horizon in a flat FRW universe. In this form, the radius of the apparent horizon R_A coincides with the Hubble horizon and is given by [65, 66]

$$R_A = \frac{1}{H}. \quad (15)$$

On the other hand, the Hawking temperature on the apparent horizon T_A as function of the radius R_A is defined as [35]

$$T_A = \frac{1}{2\pi R_A} \left(1 - \frac{\dot{R}_A}{2HR_A} \right), \quad (16)$$

where the ratio $\dot{R}_A/2HR_A < 1$, guarantees that the Hawking temperature $T_A > 0$.

From the Gibb's equation, the entropy of the universe assuming that the DM inside the apparent horizon, is given by [67]

$$T_A dS_m = dE_m + p_m dV = dE_m. \quad (17)$$

Here, $E_m = V \rho_m$ where the volume of the pressureless matter is defined as $V = 4\pi R_A^3/3$, then

$$E_m = V \rho_m = \frac{4\pi R_A^3}{3} \rho_m. \quad (18)$$

Combining Eqs.(8), (17) and (18), we find

$$T_A \dot{S}_m = 4\pi R_A^2 \rho_m \left(\dot{R}_A + H R_A \left[\frac{Q}{3H\rho_m} - 1 \right] \right), \quad (19)$$

where \dot{S}_m correspond to the time derivative of the entropy from the matter source inside the horizon. Note that in the non-interacting limit i.e., $Q = 0$ the Eq.(19) reduces to the standard Gibb's equation $T_A \dot{S}_m = 4\pi R_A^2 \rho_m (\dot{R}_A - H R_A)$. Also, we observe that the evolution of the matter entropy $T_A \dot{S}_m$ increases with the introduction of the interaction term Q .

Using Eqs.(7) and (19), we get

$$T_A \dot{S}_m = 2\pi R_A^2 [f - 6(\dot{H} + H^2 - Hd/dt)f_R] \times \left(\dot{R}_A + H R_A \left[\frac{2Q}{3H[f - 6(\dot{H} + H^2 - Hd/dt)f_R]} - 1 \right] \right). \quad (20)$$

Here, as before we note that in the limit $Q = 0$, Eq.(20) reduces to expression obtained in Ref.[48], in which $T_A \dot{S}_m = 2\pi R_A^2 [f - 6(\dot{H} + H^2 - Hd/dt)f_R] (\dot{R}_A - H R_A)$.

On the other hand, the addition of the apparent horizon entropy S_A , in the framework of $f(R)$ gravity, is given by [68, 69]

$$S_A = \frac{A f_R}{4G}, \quad (21)$$

where the area of the horizon A is defined as $A = 4\pi R_A^2$.

Taking time derivative of the above equation and considering Eq.(16), the evolution of horizon entropy, can be written as

$$T_A \dot{S}_A = 4\pi \left(1 - \frac{\dot{R}_A}{2H R_A} \right) (2 \dot{R}_A f_R + R_A \dot{f}_R). \quad (22)$$

We note that Eq.(22) coincides with the evolution of the horizon entropy $T_A \dot{S}_A$ estimated in Ref. [48]. Also, we observe that $T_A \dot{S}_A$ becomes independent of the interacting term Q .

In this form, the total entropy S_{Total} due to different contributions of the apparent horizon entropy and the matter entropy, i.e., $S_{Total} = S_A + S_m$, from Eqs.(20) and (22), becomes

$$T_A \dot{S}_{Total} = 2\pi R_A^2 \left[\left(\frac{2}{R_A^2} - \frac{\dot{R}_A}{H R_A^3} \right) (2 \dot{R}_A f_R + R_A \dot{f}_R) + [f - 6(\dot{H} + H^2 - Hd/dt)f_R] \times \left(\dot{R}_A + H R_A \left[\frac{2Q}{3H[f - 6(\dot{H} + H^2 - Hd/dt)f_R]} - 1 \right] \right) \right]. \quad (23)$$

Note that the interacting-term Q modifies the evolution of the total entropy, in which the GSL of thermodynamic, increases by a factor $4\pi R_A^3 Q/3 > 0$. Also, we note that in the special case in which $f(R) = R$, the GSL from Eq.(23) results in $T_A \dot{S}_{Total} = \pi R_A^3 [R_A \rho_m^2 + 4Q/3] > 0$. In particular, in the limit $Q = 0$ and $f(R) = R$ we obtained $T_A \dot{S}_{Total} = \pi R_A^4 \rho_m^2 > 0$ and coincides with the GSL obtained in Ref. [48] (recalled, that $8\pi G = 1$).

In the following, we will analyze analytical solutions for the GSL of thermodynamics for one specific interaction term Q and a particular $f(R)$ gravity model.

IV. AN EXAMPLE FOR Q AND $f(R)$: ANALYTICAL SOLUTIONS

Let us consider that the interaction term Q is related to the total energy density of matter and takes the form [70, 71]

$$Q = 3 c^2 H \rho_m, \quad (24)$$

where c^2 is a positive definite constant and the factor 3 was considered for mathematical convenience (for a review of Q -terms see Ref. [72]).

Inserting the interaction term Q given by Eq.(24) in the energy equation of the matter given by Eq.(8), we find

$$\rho_m = \rho_{m0} a^{-3(1-c^2)}. \quad (25)$$

On the other hand, we study the power-law $f(R)$ model, as a specific case, where

$$f(R) = \alpha R^n, \quad (26)$$

in which $0 < n < 1$ and $\alpha > 0$ are constants [17–19]. Here, n is the slope of the gravity Lagrangian and α with the dimensions taken in such a way to give $f(R)$ the correct physical dimensions. The model R^n gravity, like any $f(R)$ theory, is object to experimental constraints. In this context, in Ref. [73], the authors analyzed the gravitational lensing in R^n gravity. In Ref. [74] was studied the solar system constraints for R^n model. Also recently, the constraints on R^n gravity from precession of orbits of S2-like stars was considered in Ref. [75] (see also Refs. [74, 76–78, 80]).

In this form combining Eqs.(7), (25), and (26), we get

$$a(t) \propto t^{\frac{2n}{3(1-c^2)}}, \quad (27)$$

where the exponent in the scalar factor is $\frac{2n}{3(1-c^2)} > 1$ for guarantee an accelerated phase of the universe. Considering that $0 < n < 1$, together with the condition $\frac{2n}{3(1-c^2)} > 1$, we get that the range for the parameter c^2 , becomes $\frac{(3-2n)}{3} < c^2 < 1$.

The Hubble parameter is given by

$$H(t) = \frac{2n}{3(1-c^2)} \frac{1}{t} = \sqrt{\frac{n R}{3(4n - 3[1 - c^2])}}, \quad (28)$$

and the acceleration parameter q , from Eqs.(13) and (27) is given by $q = (3 - 2n - 3c^2)/2n$.

The total entropy due to different contributions of the apparent horizon entropy and the matter entropy from Eq.(23), can be written as

$$\begin{aligned}
T_A \dot{S}_{Total} = & 2\pi R_A^2 \left[\left(\frac{2}{R_A^2} - \frac{\dot{R}_A}{H R_A^3} \right) \times \right. \\
& \left(2 \dot{R}_A R^{n-1} + (n-1) R^{n-2} R_A \dot{R} \right) \alpha n \\
& + \alpha [R^n - 6n(\dot{H} + H^2 - H d/dt) R^{n-1}] \\
& \left. \times \left(\dot{R}_A + H R_A [c^2 - 1] \right) \right], \tag{29}
\end{aligned}$$

where $R_A = 1/H = R^{-1/2} \sqrt{\frac{3(4n-3[1-c^2])}{n}}$.

In Fig.(1) we show the evolution from the early times ($R/R_0 \rightarrow +\infty$) to the current epoch ($R/R_0 = 1$) for the effective EoS parameter w_R versus the dimensionless scalar R/R_0 , for two different values of the parameter n in the model $f(R) = \alpha R^n$. Here, R_0 is the Ricci scalar at the present epoch. In order to write down values that relate the effective EoS w_R and R/R_0 , we consider Eq.(12) together with the interaction term given by Eq.(24). Here, we note that we have not a transition from the $w_R > -1$ (quintessence) to $w_R < -1$ (phantom). In this form, the interaction $Q \propto H\rho_m$ and $f(R) \propto R^n$ gravity cannot cross the phantom divide line, as could be seen from Fig.(1). In both panels, we have used three different values of the interacting-parameter c^2 , where $\frac{(3-2n)}{3} < c^2 < 1$. In the upper panel, we have taken $\alpha = 3000$, $n = 0.1$ and in the lower panel we have used $\alpha = 0.09$ and $n = 0.9$. Also, in both panels we have used $H_0 = 72.5 \text{ Km S}^{-1} \text{ Mpc}^{-1}$ [81] and $\kappa = 1$. From the upper panel, we note that at the present epoch ($R/R_0 = 1$), for the values $c^2 = 0.94$, $c^2 = 0.97$ and $c^2 = 0.99$, we find that $w_R = -0.87$, $w_R = -0.93$ and $w_R = -0.98$, respectively. Also we observe that at early times, i.e., $R/R_0 \rightarrow \infty$ we obtain that $w_R \rightarrow -0.99$ for all values of c^2 . Additionally, we note that the effective EoS parameter w_R depends on α parameter. In particular, for values of $\alpha < 3000$ the effective EoS $w_R \sim -0.99$ and for values of $\alpha > 3000$, we get that the effective $w_R > 0$.

For the case $n = 0.9$ (lower panel), we have found that at the present epoch for the values $c^2 = 0.45$, $c^2 = 0.70$ and $c^2 = 0.99$, we get that $w_R = -0.58$, $w_R = -0.73$ and $w_R = -0.85$. At early times, we obtain that $w_R \rightarrow -0.65$, $w_R \rightarrow -0.82$ and $w_R \rightarrow -0.99$, respectively. Also, we note that for values of $\alpha < 0.09$ the effective EoS $w_R \sim -0.99$ for the value $c^2 = 0.99$, $w_R \sim -0.80$ for $c^2 = 0.70$ and for the value of $c^2 = 0.45$ corresponds to $w_R \sim -0.65$. For values of $\alpha > 0.09$ the effective EoS $w_R > 0$.

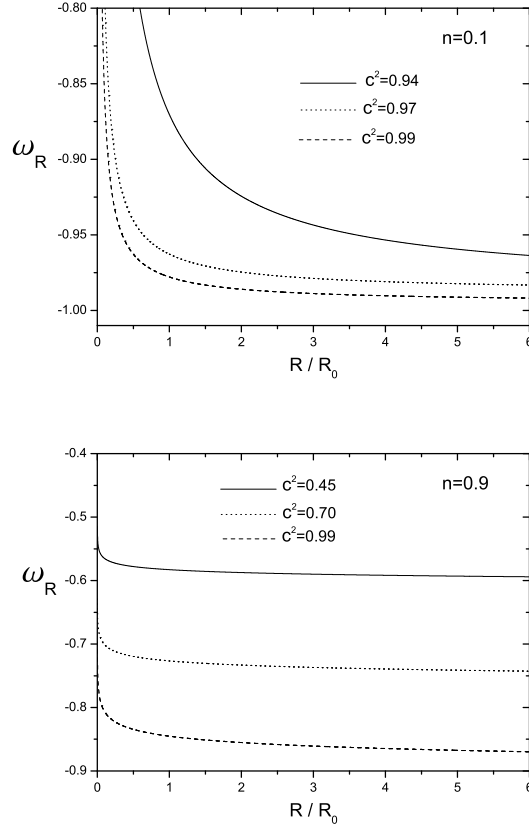


FIG. 1: Evolution of the effective EoS parameter w_R versus the dimensionless scalar R/R_0 , for two different values of the parameter n , in the model $f(R) = \alpha R^n$ and $Q = 3c^2 H \rho_m$. In both panels, we used three different values of the interacting-parameter c^2 . In the upper panel, we have taken $\alpha = 3000$, $n = 0.1$ and in the lower panel we have used $\alpha = 0.09$ and $n = 0.9$. Also, in both panels we have used $H_0 = 72.5 \text{ Km S}^{-1} \text{ Mpc}^{-1}$ and $\kappa = 1$

In Fig.(2) we represent the evolution of the GSL versus the dimensionless scalar R/R_0 , for two different values of the parameter n . In order to write down values that relate $T_A \dot{S}_{Total}$ versus R/R_0 , we considered Eq.(23). As before, in the upper panel we have used $\alpha = 3000$, $n = 0.1$ and in the lower panel we have taken $\alpha = 0.09$ and $n = 0.9$. From Fig.(2) we observe that the GSL is satisfied from the early times i.e., $R/R_0 \rightarrow \infty$ to the current epoch in which $R/R_0 = 1$. Also, in both panels we note that the GSL graphs for the value $c^2 = 0.99$ corresponds to $T_A \dot{S}_{Total} \sim 0$. Here, we observe that at early times, i.e., $R/R_0 \rightarrow \infty$ we obtain that $T_A \dot{S}_{Total} \rightarrow 0$ (adiabatic system). In particular, for $n = 0.1$ at the present time i.e., $R/R_0 = 1$ we get that $T_A \dot{S}_{Total} \simeq 47.9$ for the value $c^2 = 0.94$, $T_A \dot{S}_{Total} \simeq 6.6$

that corresponds to $c^2 = 0.97$ and $T_A \dot{S}_{Total} \simeq 1.7$ that corresponds to $c^2 = 0.99$. For the specify case $n = 0.9$ at the present time ($R/R_0 = 1$), we find that $T_A \dot{S}_{Total} \simeq 1.12$ for the value $c^2 = 0.45$, $T_A \dot{S}_{Total} \simeq 0.59$ that corresponds to $c^2 = 0.70$ and $T_A \dot{S}_{Total} \simeq 0.02$ that corresponds to $c^2 = 0.99$. Also, we note that the GSL is increased in the future i.e., $0 < R/R_0 < 1$, in which $T_A \dot{S}_{Total} > 0$.

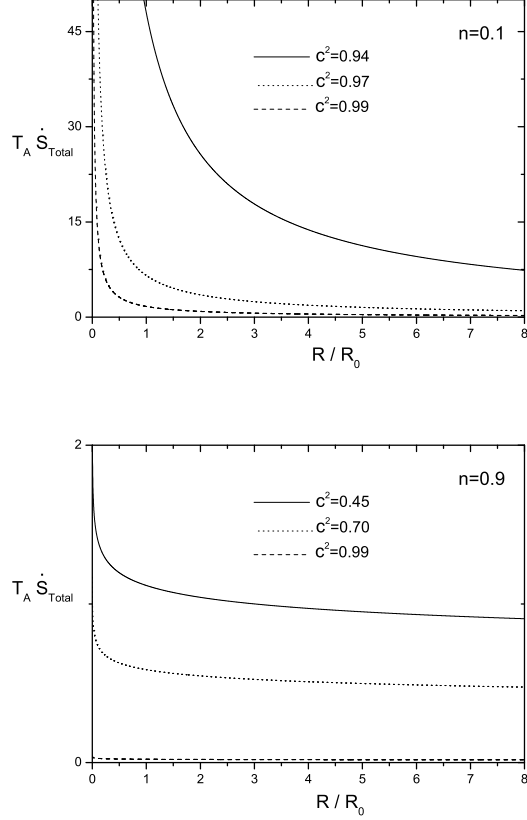


FIG. 2: Evolution of the GSL ($T_A \dot{S}_{Total}$) versus the dimensionless scalar R/R_0 , for two different values of the parameter n . As before, in both panels, we used three different values of the interaction parameter c^2 . In the upper panel, we have taken $\alpha = 3000$, $n = 0.1$ and in the lower panel we have used $\alpha = 0.09$ and $n = 0.9$. Also, in both panels we have used $H_0 = 72.5 \text{ Km S}^{-1} \text{ Mpc}^{-1}$ and $\kappa = 1$.

V. CONFORMAL TRANSFORMATION: SCALAR TENSOR GRAVITY THEORY

Since that there are two approaches to study the dark energy $f(R)$ -gravity model, is interesting to analyze the GSL of our model as a scalar tensor gravity theory, by means of a conformal transformation, specifically from the original frame (called Jordan frame) to the Einstein frame. Introducing a conformal transformation, the metric tensor $g_{\mu\nu}$ is transformed into $\tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}$, where $\Omega(x)^2$ is the conformal factor, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ represent the original and transformed metric, respectively. With this conformal transformation and together with the introduction of new field σ , defined as

$$\Omega(x)^2 = e^{\sqrt{\frac{2}{3}}\sigma} = f_R,$$

the action given by Eq.(1) becomes a Einstein Hilbert type action [89], given by

$$I_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{R(\tilde{g})}{2} - \frac{1}{2}(\tilde{\nabla}\sigma)^2 - V(\sigma) + \tilde{L}_m \right],$$

where the scalar field potential, becomes [89]

$$V(\sigma) = \text{sign}(F) \frac{(f - Rf_R)}{2f_R^2}.$$

For the case in which the metric in the new coordinates, corresponds to FRW metric, then the relation between the scale factor $\tilde{a} \equiv a_E$ in the Einstein frame and the scale factor in the Jordan frame, is given by[89]

$$a_E = e^{\frac{\sigma}{\sqrt{6}}} a, \quad (30)$$

and the time coordinate in the Einstein frame $\tilde{t} \equiv t_E$ and the time in the Jordan frame are related by the differential relationship [89]

$$e^{\frac{-\sigma}{\sqrt{6}}} dt_E = dt, \quad (31)$$

also, the transformation for the energy density of the matter in both frames, becomes

$$\tilde{\rho}_m \equiv \rho_{m_E} = \rho_m e^{-2\sqrt{\frac{2}{3}}\sigma}.$$

In the Einstein frame, the energy density ρ_E and the scalar field σ satisfy the following equations:

$$\rho'_{m_E} + 3H_E \rho_{m_E} + \sqrt{\frac{1}{6}} \sigma' \rho_{m_E} = \tilde{Q} = e^{-\frac{5}{\sqrt{6}}\sigma} Q, \quad (32)$$

$$\left[\sigma'' + 3H_E \sigma' + V_{,\sigma} - \sqrt{\frac{1}{6}} \rho_{m_E} \right] \sigma' = -\tilde{Q}. \quad (33)$$

Here, we noted that in the Einstein frame appears an effective interaction term $Q_{eff} = \tilde{Q} - \frac{1}{\sqrt{6}} \rho_{m_E} \sigma'$, and in the limit $\tilde{Q} \rightarrow 0$ (or analogously $Q \rightarrow 0$), then $Q_{eff} \rightarrow -\frac{1}{\sqrt{6}} \rho_{m_E} \sigma'$.

The Friedmann equation, in this frame, results

$$3H_E^2 = \frac{\sigma'^2}{2} + V(\sigma) + \rho_{m_E}, \quad (34)$$

where the primes denote differentiation respect to the time t_E and $H_E = a'_E/a_E$ defines the Hubble parameter in the Einstein frame.

From Eqs.(19), (22),(30) and (31), the GSL of thermodynamics due different contributions of the apparent horizon entropy and the matter entropy in the Einstein frame, can be written as

$$T_A S'_{Total} = 4\pi \left[R_A^2 \rho_{m_E} e^{\sqrt{3/2}\sigma} \left[e^{\sqrt{1/6}\sigma} R'_A + \left(\frac{\tilde{Q}}{3\rho_{m_E}(H_E - \sigma'/\sqrt{6})} - 1 \right) \right] + e^{\sqrt{2/3}\sigma} \left(1 - \frac{e^{\sqrt{1/6}\sigma} R'_A}{2} \right) \left(2R'_A + \sqrt{\frac{2}{3}} \sigma' R_A \right) \right], \quad (35)$$

where now the radius of the apparent horizon in the Einstein frame, is

$$R_A = \frac{e^{\sigma/\sqrt{6}}}{(H_E - \sigma'/\sqrt{6})}.$$

In the following, we will study the GSL of thermodynamics for our specific model. Considering the case in which $f(R)$ is given by Eq.(26) i.e., $f = \alpha R^n$ and Q by Eq.(24), we get that the scalar field potential is

$$V(\sigma) \sim \exp[-\lambda \sigma],$$

where the constant $\lambda = \sqrt{\frac{2}{3}} \left[\frac{2-n}{1-n} \right]$.

From the new equations of motion, the solution in the Einstein frame for the energy density of the matter, becomes $\rho_{m_E} \sim t_E^{-2}$, and the solution for the scalar field σ , is given by

$$\sigma = \frac{2}{\lambda} \ln(t_E). \quad (36)$$

The scale factor in the Einstein frame, by using Eqs. (27),(30) and (36), becomes

$$a_E \propto t_E^\gamma, \quad \text{where} \quad \gamma = \frac{2}{\sqrt{6}\lambda} + \left(1 - \frac{2}{\sqrt{6}\lambda} \right) \frac{2n}{3(1-c^2)}, \quad (37)$$

where the exponent in the scalar factor is $\gamma > 1$ for guarantee an accelerated phase of the universe. Using the fact that $0 < n < 1$, together with the condition $\gamma > 1$, we find that the range for the parameter c^2 , becomes $\frac{(3-2n)}{3} < c^2 < 1$, that is similar to obtained in the Jordan frame.

The Hubble parameter in the Einstein frame is given by

$$H_E = \frac{1}{a_E} \frac{da_e}{dt_E} = \frac{\gamma}{t_E} = \sqrt{\frac{\gamma}{6(2\gamma - 1)}} R_E,$$

where R_E represents the scalar curvature in the Einstein frame.

In Fig.(3) we represent the evolution of the GSL versus the dimensionless scalar ratio R_E/R_{E0} in the Einstein frame, for the case $n = 0.1$ and for two different values of the parameter c^2 . In order to write down values that relate $T_A S'_{Total}$ versus R_E/R_{E0} , we used Eq.(35), together with our specific case, i.e., $f(R) = \alpha R^n$ and $Q = 3c^2 H \rho_m$. From Fig.(3) we observe that the GSL is satisfied from the early times i.e., $R_E/R_{E0} \rightarrow \infty$ to the current epoch in which $R_E/R_{E0} = 1$. Also, we find that at early times, i.e., $R_E/R_0 \rightarrow \infty$, the GSL $T_A S'_{Total} \rightarrow 0$, for both values of c^2 . In particular, for $n = 0.1$ at the present time i.e., $R_E/R_{E0} = 1$, we get that $T_A S'_{Total} \simeq 4.3 \times 10^{-2}$, for the value $c^2 = 0.97$, and $T_A S'_{Total} \simeq 5.8 \times 10^{-2}$ that corresponds to $c^2 = 0.99$. Also, we noted that for values of the parameter $c^2 < 0.965$, the GSL of thermodynamics is negative, $T_A S'_{Total} < 0$, and then the GSL is violated for the dimensionless scalar ratio R_E/R_{E0} . For the other specify case $n = 0.9$, we find that $T_A S'_{Total} < 0$, in which the GSL is violated for the values $c^2 < 0.977$ (figure not shown). In this form, we noted that the validity of the GSL of thermodynamics in both frames is non equivalent.

VI. CONCLUSIONS

In this paper we have investigated the GSL in the context of interacting $f(R)$ gravity. We studied the GSL from the boundary of the universe to be enclosed by the dynamical apparent horizon in a flat FRW universe occupied with pressureless DM, together with the Hawking temperature on the apparent horizon. We have found that the interacting term Q modified; the curvature contributions component given by an effective EoS parameter, the deceleration parameter and the evolution of the total entropy or rather the GSL. In particular, we have obtained that the modification in the evolution of the total entropy,

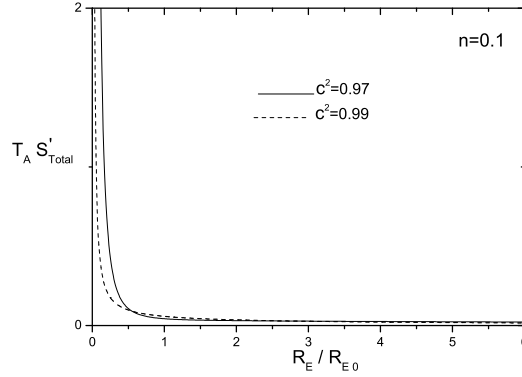


FIG. 3: Evolution of the GSL ($T_A \dot{S}'_{Total}$) versus the dimensionless scalar R_E/R_{E0} in the Einstein frame, for two different values of the parameter c^2 in the case $n = 0.1$.

results in an increases on the GSL of thermodynamics by a factor $4\pi R_A^3 Q/3 > 0$.

Our specific model is described by a model $f(R) \propto R^n$ and we have considered for simplicity the case in which the interaction term Q is related to the total energy density of matter. For this specify model, we have found analytic solutions and obtained explicit expressions for the effective EoS parameter, the deceleration parameter and the evolution of the total entropy. For this model, we observed that we do not have a transition from the value $w_R > -1$ (quintessence) to $w_R < -1$ (phantom) and the interacting $f(R) = \alpha R^n$ gravity cannot cross the phantom divide line, as could be seen from Fig.(1). Also, we have observed that the GSL is satisfied from the early times i.e., $R/R_0 \rightarrow \infty$ to the future in which GSL always $T_A \dot{S}'_{Total} > 0$.

Also, we have shown that the GSL of thermodynamics for the interacting $f(R)$ gravity is less restricted than analogous $Q = 0$ due to the introduction of a new parameter, present in the interaction term Q . In our specific model the incorporation of this parameter gives us a freedom that allows us to modify the standard $f(R)$ gravity by simply modifying the corresponding value of the parameter c^2 .

We have studied the GSL of thermodynamics for the interacting $f(R)$ in the Einstein frame through a conformal transformation. In particular, for our specific model in which $f(R) \sim R^n$ and $Q \sim c^2 H \rho_m$, we have observed that the GSL thermodynamic is violated in the Einstein frame for some values of the parameters n and c^2 . In particular, for $n = 0.1$ we have found that for values of $c^2 < 0.965$, the GSL is negative, $T_A \dot{S}'_{Total} < 0$, and then

the GSL is violated for the dimensionless ratio R_E/R_{E0} . Similarly, for $n = 0.9$ we have obtained that $T_A S'_{Total} < 0$, in which the GSL of thermodynamic is violated for the values of $c^2 < 0.977$. In this form, we have found that the validity of the GSL of thermodynamics in both frames is non equivalent.

Finally, we have not addressed other interacting- $f(R)$ models (see e.g., Refs. [82–88]). Here, a more accurate numerical calculation would be necessary for different $f(R)$ gravity models and Q interaction terms. We hope to return to this point in near future.

Acknowledgments

R.H. was supported by COMISION NACIONAL DE CIENCIAS Y TECNOLOGIA through FONDECYT grant N^o 1130628 and by DI-PUCV grant 123.724. N.V. was supported by Proyecto Beca-Doctoral CONICYT N^o 21100261.

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